Remark: The notion of ~~independents~~ independent events depends on the choice of P.

Remark: A, B ~~are~~ disjoint $\not\equiv$ A, B independent

eg: As above examples in 18.

- A, B are P-indep but A, B are not disjoint.
- $\{(1,0)\}$ and $\{(0,1)\}$ are disjoint but not P_0-indep.

$\square$

Prop: Let A, B are independent events Then

1: A, B^c are indep
2: A^c, B^c are indep

3: $P(A|B) = P(A)$.

pf 1: $\vdash$: $P(AB^c) = P(A)\,P(B^c)$

Note: $A = AB \cup AB^c$ and $(AB) \cap (AB^c) = \phi$

Then $P(A) = P(AB) + P(AB^c)$

$\Rightarrow \quad P(AB^c) = P(A)(1 - P(B)) = P(A)\,P(B^c)$.

$\square$

Def: A seq of events $(E_n)_{n=1}^{N}$ is said to be independent if it satisfies the eq:

$$P(E_{i_1} \cdots E_{i_l}) = P(E_{i_1}) \cdots P(E_{i_l})$$

for ~~any~~ any finitely many subsequence $E_{i_1}, \ldots, E_{i_l}$, where $1 \leq N \leq \infty$, and $1 \leq l < \infty$

~~e.g.~~

e.g: E_1, E_2, E_3 are independent events if we have:

$$P(E_1 E_2) = P(E_1) P(E_2)$$

$$P(E_1 E_3) = P(E_1) P(E_3)$$

$$P(E_2 E_3) = P(E_2) P(E_3)$$

and $$P(E_1 E_2 E_3) = P(E_1) P(E_2) P(E_3)$$

e.g.: Let $E_1, E_2, \ldots$ be a sequence of independent trials. Assume that the successful prob of each trial is p.

Find the prob of all trials are in successes.

Sol: Let $F_1 \equiv E_1$

$F_2 \equiv E_1 E_2$

$\vdots$

$F_n \equiv E_1 \cdots E_n$

$\vdots$

Then (F_n) is ~~decrea~~ decreasing.

Note: $\bigcap_{n=1}^{\infty} E_n = \bigcap_{n=1}^{\infty} F_n$

Hence we have

$$P(\cap E_n) = P(\cap F_n) = \lim_n P(F_n) = \lim_n P(E_1) \cdots P(E_n)$$

$$= \lim_n p^n = \begin{cases} 0 & 0 \leq p < 1 \\ 1 & p = 1 \end{cases}$$

$\square$

HW2: p105, 1, 5

p119, 4

Deadline: 3. Oct.

Ch 4. Random variables.

Throughout this chapter. (Ω, ξ, P) always denotes a prob space

Def: A function $X: \Omega \to \mathbb{R}$ is said to be a random variable if

$\{\omega \mid X(\omega) \le a\} \equiv \{X \le a\} \in \xi$. for all $a \in \mathbb{R}$

In this case, if the image of X,

$\operatorname{im} X = \{x_1, \dots x_N\}$ $(1 \le N \le \infty)$, then X is said to be a discrete random variable.

Remark:

- If $\xi = \wp(\Omega)$ then all functions in X is discrete.
- The definition of random variables depends on the choice of ξ.

e.g.: Let $\Omega = \{a, b, c\}$, $\xi \equiv \{\Omega, \phi, \{a\}, \{b, c\}\}$

Define : $X : \Omega \to \mathbb{R}$ by

$X(a)=1$, $X=(b)=2$, $X(c)=3$

Then X is not a r.v. with respect to ξ

since $\{X \le 2\} = \{a, b\} \notin \xi$

Prop: Let $X : \Omega \to \mathbb{R}$ be a function. Then the followings are equivalent:

(i) X is a r.v.

(ii) $\{X < a\} \in \xi$

(iii) $\{X > a\} \in \xi$

(iv) $\{X \ge a\} \in \xi$

pf: (i) $\Leftrightarrow$ (iii) obvious since $\{X \le a\}^c = \{X > a\}$

~~(ii) $\Leftrightarrow$ (iii)~~

(i) $\Rightarrow$ (ii) : Note: $\{X < a\} = \bigcup_{n=1}^{\infty} \{X \le a - \frac{1}{n}\}$

(ii) $\Rightarrow$ (i) : $\{X \le a\} = \bigcap_{n=1}^{\infty} \{X < a + \frac{1}{n}\}$

(ii) $\Leftrightarrow$ (iv) obvious since $\{X < a\}^c = \{X \ge a\}$

$\square$

~~Cor: If X is a r.v. then~~

Cor: If X is a r.v. then $\{X = a\} \in \xi$ $\forall a \in \mathbb{R}$

If the image of $X = \{x_1, .., x_N\}$ $(1 \le N \le \infty)$ and

$\{X = a\} \in \xi$, $\forall a \in \mathbb{R}$, then X is a r.v.

pf: Note for any $a \in \mathbb{R}$,

$$\{X = a\} = \bigcap_{n=1}^{\infty} \{a - \tfrac{1}{n} \le X \le a\}$$

$$= \bigcap_{n=1}^{\infty} \left[\{a - \tfrac{1}{n} \le X\} \cap \{X \le a\}\right] \in \xi$$

Conversely, if im $X = \{x_1, .. x_N\}$ $(1 \le N \le \infty)$ and $\{X = a\} \in \xi$, $\forall a \in \mathbb{R}$, then for any $b \in \mathbb{R}$

$$\{X \le b\} = \bigcup_{i:\, x_i \le b} \{X = x_i\} \in \xi.$$

$\square$

Prop: Let X and Y be r.v.s. Then

(i) $X \pm Y$ are r.v.s

(ii) $X \cdot Y$ is a r.v.s, $\alpha . X$ is a r.v. for all $\alpha \in \mathbb{R}$

(iii) if $(X_n)_{n=1}^{\infty}$ is a sequence of r.v.s and $X(\omega) = \lim_n X_n(\omega)$ exists for all ω, then X is a r.v.

~~Propn~~ For each subset $E \subseteq \Omega$, define

$\chi_E : \Omega \to \mathbb{R}$ by

$$\chi_E(\omega) = \begin{cases} 1 & \text{if } \omega \in E \\ 0 & \omega \notin E \end{cases}$$

χ_E is called the indicator of E.

Prop: (i) Let $E \subseteq \mathbb{R}$.
$E \in \xi$ iff χ_E is a r.v. $\boxed{}$

~~(ii) Let ...~~

HW ③ : Deadline: 17. Oct. (Fri)
P136 Ex. 4, 7

Def: Let $X: \Omega \to \mathbb{R}$ be a r.v. The distribution function (d.f) of X write F_X is

$F_X: \mathbb{R} \longrightarrow \mathbb{R}$ is defined by

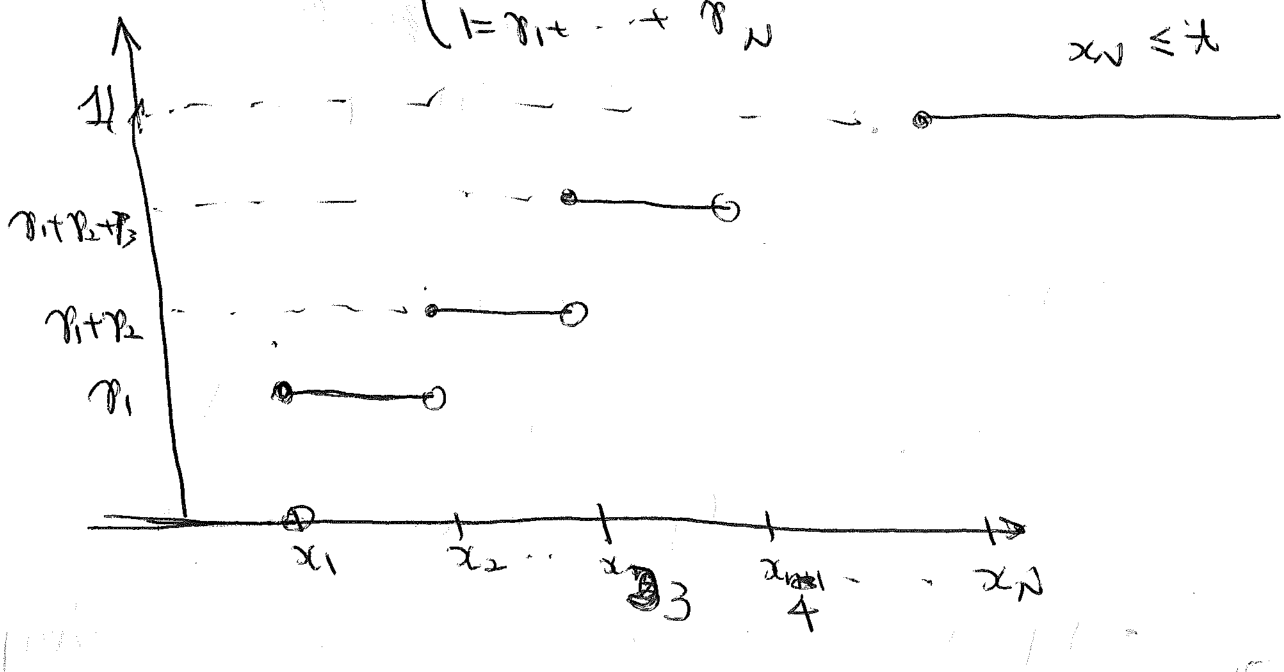
$$F_X(t) \equiv P\{X \leq t\}. \quad t \in \mathbb{R}.$$

e.g: Suppose that X is a discrete r.v with finitely many values $\{x_1, \dots, x_N\}$, $(x_1 < x_2 < \dots < x_N)$

Let $p_i = P\{X = x_i\}$

Then

$$F_X(t) \equiv \begin{cases} 0 & t < x_1 \\ p_1 & x_1 \leq t < x_2 \\ p_1 + p_2 & x_2 \leq t < x_3 \\ p_1 + p_2 + \dots + p_n & x_n \leq t < x_{n+1} \\ 1 = p_1 + \dots + p_N & x_N \leq t \end{cases}$$



Prop: With the notation as above, then we have

(i) ~~F_X~~ $0 \le F_X(t) \le 1$, for all $t \in \mathbb{R}$

(ii) $F_X(t)$ is non-decreasing, i.e: if $t_1 \le t_2$ then $F_X(t_1) \le F_X(t_2)$

(iii) $\lim_{t \to \infty} F_X(t) = 1$, $\lim_{t \to -\infty} F_X(t) = 0$

(iv) $F_X(t)$ is right continuous, i.e: $\forall\, t_0 \in \mathbb{R}$

$\exists$ a seq $t_0 < x_n$, with $x_n \downarrow t_0$, then

$F_X(x_n)$ ~~$F_X(x_n)$~~ $\to F_X(t_0)$ as $n \to \infty$

pf (iii): F_1: $\lim_{t \to \infty} F_X(t) = 1$

~~HKi~~ Need to show: If (t_n) is any seq with $t_n \uparrow \infty$, then $F_X(t_n) \to 1$

pf: Since $t_n \uparrow$, $\{X \le t_n\} \uparrow$

~~Now~~ and $\Omega = \bigcup_{n=1}^{\infty} \{X \le t_n\}$,

$$1 = P(\Omega) = \lim_n P\{X \le t_n\} = \lim_n F_X(t_n)$$

$\square\square$